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Fractional Fick's law: the direct way

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Abstract

Lévy flights, which are Markovian continuous time random walks possibly accounting for extreme events, serve frequently as small-scale models for the spreading of matter in heterogeneous media. Among them, Brownian motion is a particular case where Fick's law holds: for a cloud of walkers, the flux is proportional to the gradient of the probability density of finding a particle at some place. Lévy flights resemble Brownian motion, except that jump lengths are distributed according to an α-stable Lévy law, possibly showing heavy tails and skewness. For α between 1 and 2, a fractional form of Fick's law is known to hold in infinite media: that the flux is proportional to a combination of fractional derivatives or the order of $\alpha - 1$ of the density of walkers was obtained as a consequence of a fractional dispersion equation. We present a direct and natural proof of this result, based upon a novel definition of usual fractional derivatives, involving a convolution and a limiting process. Taking account of the thus obtained fractional Fick's law yields fractional dispersion equation for smooth densities. The method adapts to domains, limited by boundaries possibly implying non-trivial modifications to this equation.

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1. Introduction

Anomalous spreading occurs 'ubiquitously in Nature' [1], within the framework of processes which may be Markovian or not. Here, disregarding memory effects, we focus on the first case, and more especially on the spreading of matter in heterogeneous media when it shows heavy tails, not compatible with the notion of a mean-square displacement. A seminal example, in the field of charge transport, was illustrated in [2, 3], and many others followed in various domains of physics, biology or economics [4–6]. Among them environmental sciences are well represented, with many data for solute spreading in underground porous media [7, 8], conveniently described by space-fractional dispersion equations [9–12].

A celebrated family of continuous time random walks, called Lévy flights, serves as a small-scale model for anomalous transport (of charge in semiconductors or of a solute in sand columns or in aquifers among many examples) in this context. The correspondence between Lévy flights and space-fractional dispersion equations was thoroughly studied [13–23]. And a fractional variant of Fick's law was shown to hold [24] in infinite media for the concentration of a solute or any cloud of particles, spreading according to a space fractional dispersion equation, thus confirming an intuition of [25]. Also starting from a dispersion equation involving fractional derivatives with respect to space and time, the relationship between flux and concentration has been addressed in [27], in the words of underground transport. And the flux of particles performing a special class of random walks with memory and Gaussian jump length has been computed in [26].

In semi-infinite or bounded media, it may be necessary to modify space fractional dispersion equations [28–30], depending on the physical properties of the boundaries limiting the domain. This is due to the non-local character of fractional derivatives. For symmetric Lévy flights in a half-space limited by a reflective barrier, the result was stated by considering the even extension of the density of spreading particles, then making use of Fourier's analysis. Nevertheless, many practical situations involve boundaries and also other possible causes of skewness. Stable laws, which are tightly associated with non-normal spreading, may be skewed, and disregarding this possibility would desprive us of a significant part of the possibilities, offered by fractional dispersion equations. Hence a tool, able to model boundary data and skewed Lévy flights, is highly desirable. Moreover, it is allowed to hope that directly connecting flux and density may give more insight into practical situations, possibly involving singularities due to sources or sinks.

In fact, counting particles crossing a definite location per unit of time yields the flux corresponding to a random walk, in an infinite, semi-infinite or bounded medium, for a given concentration profile. In general, measurements correspond to time and length scales much larger than those of particle motions, which we therefore let tend to zero. We present here a novel expression for the left inverse of Riemann–Liouville's fractional integrals [31, 32], which allows us to address the diffusive limit of the operator, mapping concentration to small-scale flux, for Lévy flights. With this tool, we follow a route, different from that of many authors [13–23] who studied the diffusive limit of the concentration for specified initial data and, then, noted that it satisfies a space-fractional dispersion equation. We state a fractional variant of Fick's law, valid for possibly non-smooth concentration profiles which may occur if there are sources. The method will adapt to problems with boundaries more easily than the usual route based upon Fourier's analysis, that is well suited for infinite media but not comfortable when there are boundaries. Combined with mass conservation principle, the fractional Fick's law results in equations, ruling the evolution of the density of a cloud of walkers. For the moment, we concentrate on the dimension one.

To do this, we first note that, for Lévy flights the flux computed on the scale of particle motion, depends on the parameters of the random walk and on the concentration at considered time. For a Markovian process, such as Lévy flights, the latter resumes the influence of the past. This includes initial data, which do not need being specified in our reasoning, except when we take examples. The passage to the macroscopic scale is then performed by taking the limit of the operator mapping flux to concentration, when time and length scales of the random walk tend to zero while satisfying some scaling law, analogous to that used in [20, 21]. A novel expression for the left inverse of Riemann–Liouville fractional integrals is stated, which allows us to see that the above-mentioned limit combines two kinds of fractional derivatives. We thus arrive at a fractional generalization of Fick's law for symmetric or skewed Lévy flights in infinite media or with a reflective boundary. From this we deduce the fractional dispersion equation for the evolution of the density of particles, and already known results are retrieved, for the diffusive limit of random walks in infinite media or in domains with boundaries under symmetry assumptions. Less classical situations for skewed Lévy flights with a boundary are considered, and illustrated by direct Monte Carlo simulations.

2. The flux of particles performing Lévy flights

For particles performing a Markovian random walk, the flux depends on the concentration and on the transition probability density function. It also depends on the geometry of the medium and on the physical properties of the boundaries, if there are, as we will see. After having set these points, we will focus on what happens to the flux when time and length scales of the random walk are made small, compared with those, characterizing the variations of the density of the cloud of particles.

2.1. Physical setting

Consider a cloud of independent walkers, performing Lévy flights: each of them makes successive instantaneous jump, separated by pausing times. Jump amplitudes are independent variables, with density $\varphi_l(x) = \frac{1}{l} L_\alpha^\theta\left(\frac{x}{l}\right)$. Here, L_α^θ denotes the density of a normalized Lévy law of stability index α and skewness parameter θ [33–35], whose essential properties are recalled in appendix A. Waiting times are also independent, and for the sake of simplicity we assume here that they are distributed according to the density $\psi_\tau(t) = \frac{1}{\tau}\,\mathrm{e}^{-\frac{t}{\tau}}$, though more general possibilities were considered by [36]. Here, τ is the mean waiting time and l is a length scale.

Let $P(x,t)$ be the density of the probability of finding a walker in $[x, x + \mathrm{d}x]$ at time t. The flux through the abscissa x is the balance of particles crossing x to the right or to the left during $[t, t + \mathrm{d}t[$, divided by $\mathrm{d}t$. Moreover, the probability of making no jump during time interval $[0, t]$ is $\mathrm{e}^{-\frac{t}{\tau}}$. Hence, the probability of making exactly one jump during $[0, t]$ is $\tau^{-1}\int_0^t \mathrm{e}^{-\frac{t'}{\tau}}\,\mathrm{e}^{-\frac{t-t'}{\tau}}\,\mathrm{d}t' = \frac{t}{\tau}\,\mathrm{e}^{-\frac{t}{\tau}}$. During $[t, t + \mathrm{d}t]$ or $[0, \mathrm{d}t]$, we obtain $\frac{\mathrm{d}t}{\tau}$. Therefore, the probability of making one jump during infinitesimal time interval $[t, t + \mathrm{d}t[$ is $\frac{\mathrm{d}t}{\tau}$. And the probability of making more than one jump is of the order of $\mathrm{d}t^2$.

Hence, when the random walk takes place in an infinite medium, the flux is equal to

$$
{}_\infty\mathcal{W}_{l,\tau}^{\alpha,\theta}(x,t)P = \tau^{-1}\left[\int_0^\infty P(x-y,t)F_{\alpha,\theta}^{+}\left(\frac{y}{l}\right)\mathrm{d}y - \int_0^\infty P(x+y,t)F_{\alpha,\theta}^{-}\left(\frac{-y}{l}\right)\mathrm{d}y\right], \tag{1}
$$

with $F_{\alpha,\theta}^{+}(y/l)$ being the probability $\int_y^{+\infty}\frac{1}{l}L_{\alpha,\theta}(z/l)\,\mathrm{d}z = \int_{y/l}^{+\infty}L_{\alpha,\theta}(z)\,\mathrm{d}z$ for a jump to be to the right while having an amplitude of more than y. Similarly, $F_{\alpha,\theta}^{-}(-y/l)$ is the probability $\int_{-\infty}^{-y}\frac{1}{l}L_{\alpha,\theta}(z/l)\,\mathrm{d}z$ for a jump to have a modulus of more than y, but to the left.

The expression giving the flux may be modified more or less deeply by the presence of a boundary, depending on whether it is allowed to release particles or not [37, 38]. Two examples illustrate this point.

2.2. With a boundary

We may imagine an absorbing boundary [38] such that walkers are killed when hitting the wall: after that they no longer contribute to the random walk. Then, expression (1) holds, with HP in place of P in the right-hand side. Here H denotes *Heavisde's function*, if the wall is located at $x = 0$, as will be supposed here.

Oppositely, imagine that each particle hitting a purely reflecting wall (here located at $x = 0$) bounces and finally flies the length of the jump, which had been assigned to it before the shock, but remains on the same side ($x > 0$) of the wall [28]. In this case, we have to take account of two points when counting particles crossing x to the left or to the right. First, jumps directed to the left and starting from $x + y$ ($y > 0$) arrive at the right of x hence do not enter the balance if the amplitude is larger than $2x + y$. Second, jumps directed to the left and starting from $x - y$, with $0 < y < x$, may cross x to the right if the amplitude is of more than $2x - y$. Consequently, in this case the flux is given by the mapping ${}_0\mathcal{W}^{\alpha,\theta}_{l,\tau}(x, t)$ defined by

$$
\begin{aligned}
{}_0\mathcal{W}^{\alpha,\theta}_{l,\tau}(x,t)P = \tau^{-1}\bigg[\int_0^x P(x-y,t)\left[F^+_{\alpha,\theta}\left(\frac{y}{l}\right) + F^-_{\alpha,\theta}\left(\frac{y-2x}{l}\right)\right]\mathrm{d}y \\
- \int_0^\infty P(x+y,t)\left[F^+_{\alpha,\theta}\left(\frac{-y}{l}\right) - F^-_{\alpha,\theta}\left(\frac{-2x-y}{l}\right)\right]\mathrm{d}y\bigg].
\end{aligned}
$$

Setting $P^*(x, t) = P(x, t)$ for $x > 0$ and $P^*(x, t) = P(-x, t)$ for $x < 0$ (even extension of P) we obtain

$$
\begin{aligned}
{}_0\mathcal{W}^{\alpha,\theta}_{l,\tau}(x,t)P = \tau^{-1}\int_0^\infty P^*(x-y,t)F^+_{\alpha,\theta}\left(\frac{y}{l}\right)\mathrm{d}y - \tau^{-1}\int_0^\infty P(x+y,t)F^-_{\alpha,\theta}\left(\frac{-y}{l}\right)\mathrm{d}y \\
- \tau^{-1}\int_x^\infty P^*(x-y,t)\left(F^+_{\alpha,\theta}\left(\frac{y}{l}\right) - F^-_{\alpha,\theta}\left(\frac{-y}{l}\right)\right)\mathrm{d}y. \qquad (2)
\end{aligned}
$$

2.3. Taking the macroscopic limit

Let us look at what happens to operators ${}_\infty\mathcal{W}^{\alpha,\theta}_{l,\tau}(x, t)$ and ${}_0\mathcal{W}^{\alpha,\theta}_{l,\tau}(x, t)$ when we let l and τ become small, compared with measurements scales. Fractional dispersion equations were shown to hold in the diffusive limit provided the *scaling* $l^\alpha/\tau = K$ [18, 20, 21] holds, which implies $\tau^{-1} = Kl^{-\alpha}$ in (1) and (2). We address the limiting behaviour of mappings ${}_\infty\mathcal{W}^{\alpha,\theta}_{l,\tau}(x, t)$ and ${}_0\mathcal{W}^{\alpha,\theta}_{l,\tau}(x, t)$ in this context.

In expressions (1) and (2), we therefore will take the limit of differences between terms of the form of $l^{-\alpha}\int_0^\infty P(x \mp y, t)F^\pm_{\alpha,\theta}\left(\frac{\pm y}{l}\right)\mathrm{d}y$ when l tends to zero, while the function P remains fixed. Let us show that the limit combines derivatives of the order of $\alpha - 1$.

3. A novel fractional tool

That Lévy statistics correspond to fractional partial differential equations is now well known, since stable densities are fundamental solutions [39]. We will see that in the diffusive limit the flux of a cloud of particles performing one-dimensional Lévy flights can be thought of as being a combination of fractional derivatives of the density of walkers. In this purpose, let us first see that taking the limit of certain convolution yields a fractional derivative.

3.1. A novel expression for the left inverse of Riemann–Liouville's integrals

Riemann–Liouville and Marchaud's derivatives of the order of α' between 0 and 1 are defined in appendix B by explicit formulae. Marchaud's right-sided derivative $D^{\alpha'}_-$ yields the left inverse of right-sided integrals $I_-^{\alpha'}$ in $I_-^{\alpha'}(L^p]-\infty,a])$, with p between 1 and $1/\alpha'$ [31]. Riemann–Liouville's derivative $\mathcal{D}_-^{\alpha'}$ computes the left inverse of right-sided integrals $I_-^{\alpha'}$ in $I_-^{\alpha'}(L^p[a,+\infty[\cap L^1[a,+\infty[)$ only [32, 40]. Hence, both right-sided derivatives coincide in $I_+^{\alpha'}(L^p[a,+\infty[\cap L^1[a,+\infty[)$. Marchaud's definition allows us to consider functions not tending to zero very rapidly near infinity, but needs some care in view of the singularity at point x. We will see that the limit, when l tends to zero, of $l^{-\alpha}\int_0^\infty f(x\pm y)F\left(\frac{\pm y}{l}\right)\mathrm{d}y$ is the left inverse of $I_\pm^{\alpha-1}$, hence a derivative of order $\alpha-1$.

Not surprisingly, the claimed result holds under definite assumptions for F. To be more precise, we will say that F, integrable over $[0,+\infty[$, satisfies hypothesis H_1 if $\int_0^\infty F(y)\,\mathrm{d}y=0$. For α between 1 and 2, it satisfies hypothesis $H_2(\alpha)$ if in a neighbourhood of $+\infty$ it is of the form $F_1(y)+Cy^{-\alpha}$, with F_1 and $F_1(y)y^{\alpha-1}$ being integrable. With these notations, we have the following theorem:

Theorem. *Let F satisfy H_1 and $H_2(\alpha)$, with α in $]1,2[$. Let p satisfy $1\leqslant p<1/(\alpha-1)$.*

(i) Then, for f in $I_-^{\alpha-1}L^p[a,+\infty[$, the limit (in $L^p[a,+\infty[$) of $l^{-\alpha}\int_0^\infty f(x+y)F\left(\frac{y}{l}\right)\mathrm{d}y$ when l tends to zero is $\int_0^{+\infty}I_+^{\alpha-1}(HF)(s)\,\mathrm{d}s\times D_-^{\alpha-1}f(x+)$.

(ii) For f in $I_+^{\alpha-1}L^p]-\infty,a]$, the limit (in $L^p]-\infty,a]$) of $l^{-\alpha}\int_0^\infty f(x-y)F\left(\frac{y}{l}\right)\mathrm{d}y$ when l tends to zero is $\int_0^{+\infty}I_+^{\alpha-1}(HF)(s)\,\mathrm{d}s\times D_+^{\alpha-1}f(x-)$.

Proof. It is enough to prove (i), identical arguments serve for (ii). □

Due to theorem 6.1 of [31], from $f\in I_-^{\alpha-1}L^p[a,+\infty[$ we deduce that we have $f(x)=I_-^{\alpha-1}\varphi(x)$ with $\varphi(x)=D_-^{\alpha-1}f(x)$ (in $L^p[a,+\infty[$) being Marchaud's derivative of f. Hence, in order to prove point (i), let us compare function φ and the limit of $l^{-\alpha}\int_0^\infty f(.+y)F\left(\frac{y}{l}\right)\mathrm{d}y$ under assumptions $H_1-H_2(\alpha)$. We have

$$l^{-\alpha}\int_0^\infty \left(I_-^{\alpha-1}\varphi\right)(x+t)F\left(\frac{t}{l}\right)\mathrm{d}t=\frac{l^{-\alpha}}{\Gamma(\alpha-1)}\int_0^{+\infty}F\left(\frac{t}{l}\right)\int_{x+t}^{+\infty}\varphi(y)(y-x-t)^{\alpha-2}\,\mathrm{d}y\,\mathrm{d}t.$$

Setting $t=lT$ and then $y=x+ls$ in the right-hand side, we obtain that the above expression is equal to

$$\frac{1}{\Gamma(\alpha-1)}\int_0^{+\infty}F(T)\int_T^{+\infty}\varphi(x+ls)(s-T)^{\alpha-2}\,\mathrm{d}s\,\mathrm{d}T.$$

Then, Fubini's theorem yields

$$l^{-\alpha}\int_0^\infty \left(I_-^{\alpha-1}\varphi\right)(x+t)F\left(\frac{t}{l}\right)\mathrm{d}t=\frac{1}{\Gamma(\alpha-1)}\int_0^{+\infty}\varphi(x+ls)\int_0^s F(T)(s-T)^{\alpha-2}\,\mathrm{d}T\,\mathrm{d}s, \tag{3}$$

as soon as $I_+^{\alpha-1}(HF)(s)=\frac{1}{\Gamma(\alpha-1)}\int_0^s F(T)(s-T)^{\alpha-2}\,\mathrm{d}T$ is integrable in R^+. This point follows from Lemma below, proved in appendix D.

Lemma. *If F satisfies H_1 and $H_2(\alpha)$, with $1<\alpha<2$, then $\int_0^s F(T)(s-T)^{\alpha-2}\,\mathrm{d}T$ is integrable in R^+.*

In the right-hand side of (3) we have $\int_R\varphi(x+ls)\left(I_+^{\alpha-1}(HF)\right)(s)\,\mathrm{d}s$ which, by theorem 1.3 of [31], is an approximation to $\int_0^{+\infty}I_+^{\alpha-1}(HF)(s)\,\mathrm{d}s$ times identity in L^p due to lemma.

For φ in $L^p]-\infty,a]$, instead of (3) we have

$$l^{-\alpha}\int_0^\infty \left(I_-^{\alpha-1}\varphi\right)(x+t)F\left(\frac{t}{l}\right)\mathrm{d}t=\frac{1}{\Gamma(\alpha)}\int_0^{+\infty}\varphi(x-ls)\int_0^s F(T)(s-T)^{\alpha-2}\,\mathrm{d}T\,\mathrm{d}s. \tag{4}$$

Hence the above theorem holds. It states that the limits of $l^{-\alpha}\int_0^\infty f(x\pm y)F\left(\frac{y}{l}\right)\mathrm{d}y = l^{-\alpha+1}\int_0^\infty f(x\pm ly)F(y)\,\mathrm{d}y$ coincide with left- and right-sided Marchaud's derivatives. Formally setting $F = \Sigma_{j\geqslant 0}(-1)^j\binom{\alpha-1}{j}\delta_j$ would retrieve Grüunwald–Letnikov approximations to Marchaud's derivatives [31].

Let us now show that convolutions, akin to those of the theorem, appear when we compute a flux in the context of Lévy flights.

3.2. Back to the flux

Expressions of the form of $l^{-\alpha}\int_0^\infty f(x\pm y)F^{\mp}_{\alpha,\theta}\left(\frac{\mp y}{l}\right)\mathrm{d}y$ are present on the right-hand sides of (1) and (2) giving the flux. Nevertheless, cumulated probabilities $F^-_{\alpha,\theta}(-.)$ and $F^+_{\alpha,\theta}$ satisfy $H_2(\alpha)$, but of course not H_1. In fact, it is possible to find some function $f_{\alpha,\theta}$ such that $\tilde F^-_{\alpha,\theta}$ and $\tilde F^+_{\alpha,\theta}$, defined by setting $\tilde F^-_{\alpha,\theta}(-y) = F^-_{\alpha,\theta}(-y) - f_{\alpha,\theta}(y)$ and $\tilde F^+_{\alpha,\theta}(y) = F^+_{\alpha,\theta}(y) - f_{\alpha,\theta}(y)$, satisfy H_1. In order to keep with functions in L^1, also satisfying $H_2(\alpha)$, it is enough to take for $f_{\alpha,\theta}$ any integrable function, compactly supported (e.g. in [0, 1]) and whose integral $\int_0^{+\infty} f_{\alpha,\theta}(y)\,\mathrm{d}y$ is equal to $\mathcal{I}_{\alpha,\theta} = \int_0^{+\infty} F^-_{\alpha,\theta}(-y)\,\mathrm{d}y = \int_0^{+\infty} F^+_{\alpha,\theta}(y)\,\mathrm{d}y$, which is possible due to (E.2). With these notations, since scaling $\tau^{-1} = Kl^{-\alpha}$ holds, the second integral on the right-hand side of (1), $Kl^{-\alpha}\int_0^\infty P(x+y,t)F^-_{\alpha,\theta}\left(\frac{-y}{l}\right)\mathrm{d}y$, is equal to

$$Kl^{1-\alpha}P(x,t)\mathcal{I}_{\alpha,\theta} + Kl^{-\alpha}\int_0^\infty P(x+y,t)\tilde F^-_{\alpha,\theta}\left(\frac{-y}{l}\right)\mathrm{d}y + Kl^{-\alpha}\int_0^\infty (P(x+y,t)-P(x,t))f_{\alpha,\theta}\left(\frac{y}{l}\right)\mathrm{d}y. \tag{5}$$

Similarly, the first expression in (1), $Kl^{-\alpha}\int_0^\infty P(x-y,t)F^+_{\alpha,\theta}\left(\frac{y}{l}\right)\mathrm{d}y$, is equal to

$$Kl^{1-\alpha}P(x,t)\mathcal{I}_{\alpha,\theta} + Kl^{-\alpha}\int_0^\infty P(x-y,t)\tilde F^+_{\alpha,\theta}\left(\frac{y}{l}\right)\mathrm{d}y + Kl^{-\alpha}\int_0^\infty (P(x-y,t)-P(x,t))f_{\alpha,\theta}\left(\frac{y}{l}\right)\mathrm{d}y. \tag{6}$$

Expressions containing $l^{-\alpha}P(x,t)\mathcal{I}_{\alpha,\theta}$ in (1) or (2) cancel each other when we take the difference between (6) and (5). Moreover, appropriately choosing $f_{\alpha,\theta}$ yields that the limits of $l^{-\alpha}\int_0^\infty (P(x\pm y,t)-P(x,t))f_{\alpha,\theta}\left(\frac{y}{l}\right)\mathrm{d}y$ (where P is fixed) are Kolwankar and Gangal's local fractional derivatives [41], which are less currently used than those of Riemann, Liouville, Marchaud, Grüunwald and Letnikov, and are rapidly described in appendix C.

For $\alpha < 2$, the appropriate choice of $f_{\alpha,\theta}$ is $f_{\alpha,\theta}(t) = \mathcal{I}_{\alpha,\theta}(2-\alpha)(1-t)^{1-\alpha}\chi_{[0,1]}$; then, we have $l^{-\alpha}\int_0^{+\infty} f_{\alpha,\theta}(y/l)(P(x+y,t)-P(x,t))\,\mathrm{d}y = \mathcal{I}_{\alpha,\theta}l^{1-\alpha}(2-\alpha)\int_0^1(1-s)^{1-\alpha}(P(x+ls,t)-P(x,t))\,\mathrm{d}s$. Consequently, when $P(.,t)$ has a local derivative (w.r.t. space) of order $\alpha-1$ at point $x+$, $l^{-\alpha}\int_0^{+\infty} f_{\alpha,\theta}(y/l)(P(x+y,t)-P(x,t))\,\mathrm{d}y$ has a limit when l tends to zero, and the limit is $\mathcal{I}_{\alpha,\theta}\Gamma(3-\alpha)$ times the right-sided local derivative $D_+^{\mathrm{KG},\alpha-1}P(x,t)$ of order $\alpha-1$. The same holds at the left of x: $l^{-\alpha}\int_0^{+\infty} f_{\alpha,\theta}(y/l)(P(x-y,t)-P(x,t))\,\mathrm{d}y$ tends to $-\mathcal{I}_{\alpha,\theta}\Gamma(3-\alpha)D_-^{\mathrm{KG},\alpha-1}P(x,t)$. When f is differentiable at x, $D^{\mathrm{KG},\alpha-1}\pm P(x,t)$ and the limits are equal to zero.

For $\alpha = 2$, we have usual derivatives instead of fractional ones, the above reasoning is no longer relevant, and we postpone to the end of subsection 4.2 the proof that the method yields Fick's law simply and directly. For α strictly between 1 and 2, let us now put together the pieces of the right-hand sides of (5) and (6) in the limit l tending to zero.

4. Fractional Fick's law and dispersion equation

4.1. Limits of the right-hand sides of (5) and (6)

The right-hand sides of (1) and (2), computing the flux of particles from the density $P(.,t)$, are obtained from expressions such as (5) and (6). Let us assume that $P(.,t)$ has Marchaud's and Kolwankar–Gangal's derivatives of order $\alpha-1$, which are integrable, bounded and continuous. The first-order derivative may not exist everywhere, thus allowing for singularities.

Then, on the right-hand sides of (5) and (6), at least for α strictly between 1 and 2 (for the moment), $l^{-\alpha}$ times the first integrals has a limit in L^p which also holds pointwise. The limit is $\lambda_+ D_+^{\alpha-1}P(x,t)$ for $l^{-\alpha}\int_0^{\infty}P(x+y)\tilde F^-_{\alpha,\theta}\left(\frac{-y}{l}\right)\mathrm{d}y$ in (5) and $\lambda_- D_-^{\alpha-1}P(x,t)$ for $l^{-\alpha}\int_0^{\infty}P(x-y)\tilde F^+_{\alpha,\theta}\left(\frac{y}{l}\right)\mathrm{d}y$ in (6), with

$$\lambda_+=\int_0^{+\infty}I_+^{\alpha-1}H\tilde F^+_{\alpha,\theta}(y)\,\mathrm{d}y,\ \lambda_-=\int_0^{+\infty}I_+^{\alpha-1}H\tilde F^-_{\alpha,\theta}(-y)\mathrm{d}y. \tag{7}$$

For integrals $Kl^{-\alpha}\int_0^{+\infty}f_{\alpha,\theta}(y/l)(P(x\pm y,t)-f(x))\,\mathrm{d}y$, the limit is

$$\mp K\mathcal{I}_{\alpha,\theta}D_\pm^{\mathrm{KG},\alpha-1}P(x,t),$$

which is zero on intervals $[a,b]$ where $P(.,t)$ belongs to Hüolder spaces $H^{\alpha-1+\varepsilon}[a,b]$, *a fortiori* if $P(.,t)$ is derivable. For parameter $\mathcal{I}_{\alpha,\theta}$, we have the exact expression (E.2), proved in appendix E. It agrees with numerical integrations, based upon integral expressions [42] for stable Lévy distributions.

Oppositely, computing $\lambda_\pm$ from (7) is not easy, but can be avoided by comparing $D_\pm^{\alpha-1}f$ against the limit of $l^{-\alpha}\int_0^{\infty}(f(x\pm y)-f(x))F^{\mp}_{\alpha,\theta}\left(\frac{\mp y}{l}\right)\mathrm{d}y$ for some particular function f, provided it belongs to $I_+^{\alpha-1}(L^p]-\infty,a])$ or $I_-^{\alpha-1}(L^p[a,+\infty[)$. Note that the latter holds provided the Marchaud's derivative of order $\alpha-1$ belongs to $L^p]-\infty,a]$, or $L^p[a,+\infty[$, respectively. The comparison will be simpler with functions f whose local derivative of order $\alpha-1$ is identically zero in neighbourhoods of infinity.

To achieve this, consider $f=\chi_{[1,2[}$ and compute λ_-. Indeed, for x in $]1,2[$, the local derivative exists and is equal to zero, while we have $l^{-\alpha}\int_0^{+\infty}(f(x+y)-f(x))F^-_{\alpha,\theta}(-y/l)\,\mathrm{d}y=l^{-\alpha}\int_{2-x}^{+\infty}F^-_{\alpha,\theta}(-y/l)\,\mathrm{d}y=C_{\alpha,-\theta}\frac{(2-x)^{1-\alpha}}{\alpha-1}+O(l)$ for $1<\alpha<2$, with $C_{\alpha,-\theta}$ being defined by (A.2). We also have $D_-^{\alpha-1}\chi_{[1,2[}(x)=\frac{-1}{\Gamma(1-\alpha)}\int_{2-x}^{+\infty}y^{-\alpha}\,\mathrm{d}y=\frac{(2-x)^{1-\alpha}}{\Gamma(2-\alpha)}$. For $x\geqslant 2$, due to $D_-^{\alpha-1}\chi_{[1,2[}(x)=0$, $D_-^{\alpha-1}\chi_{[1,2[}$ belongs to $L^p[1,+\infty[$ for $1\leqslant p<\frac{1}{\alpha-1}$, hence $\chi_{[1,2[}$ belongs to $I_-^{\alpha-1}L^p[1,+\infty[$, according to theorem 6.2 of [31]. That $f(x)=I^{\alpha-1}_{2,-}\left(\frac{(2-x)^{1-\alpha}}{\Gamma(2-\alpha)}\right)$ holds also can be checked directly with pen and paper. Hence, we have $\lambda_-=-\frac{\Gamma(2-\alpha)}{\alpha-1}C_{\alpha,-\theta}=\frac{\sin\frac{\pi}{2}(\alpha+\theta)}{\sin\pi\alpha}$, and similarly $\lambda_+=\frac{\sin\frac{\pi}{2}(\alpha-\theta)}{\sin\pi\alpha}$.

4.2. Fractional Fick's law

For sufficiently well-behaved functions (in $L^p(R)\cap H^{\alpha-1+\varepsilon}(R)$) the Kolwankar and Gangal's derivative exists and is identically zero. Then, the limit $Q(x,t)$ of mapping 'flux through x' ${}_\infty\mathcal{W}^{\alpha,\theta}_{l,\tau}(x,t)$, given by (1) for random walks in unbounded domains, is

$$f\mapsto K\left(\frac{\sin\frac{\pi}{2}(\alpha-\theta)}{\sin\pi\alpha}D_+^{\alpha-1}f(x)-\frac{\sin\frac{\pi}{2}(\alpha+\theta)}{\sin\pi\alpha}D_-^{\alpha-1}f(x)\right), \tag{8}$$

in agreement with [24]. We know from [20] that, when τ and l tend to zero while satisfying the it scaling $l^\alpha/\tau=K$, the concentration of a cloud of particles performing Lévy flights in an infinite medium without any source or sink tends to a limit, which can be computed from Laplace–Fourier analysis. The thus obtained density is derivable, hence (8) holds. Nevertheless, even is an infinite medium, possibly existing sources can be incorporated into the above reasoning, which only uses the dynamics of particles once they have been launched in the medium and run the random walk. Thus introduced singularities may result in densities that are not derivable near sources. For such situations we need an expression of the macroscopic flux, valid for concentrations, which may fail to be derivable at some points. In an infinite medium, it is given by mapping $Q(x,t)$ with

$$Q(x,t)P = K\left(\frac{\sin\frac{\pi}{2}(\alpha-\theta)}{\sin\pi\alpha}D_{+}^{\alpha-1}P(x,t) - \frac{\sin\frac{\pi}{2}(\alpha+\theta)}{\sin\pi\alpha}D_{-}^{\alpha-1}P(x,t)\right) - K\mathcal{I}_{\alpha,\theta}\Gamma(3-\alpha)\left[D_{+}^{\mathrm{KG},\alpha-1}P(x,t) + D_{-}^{\mathrm{KG},\alpha-1}P(x,t)\right], \tag{9}$$

which is a fractional variant of Fick's law, slightly more general than (8). Since Fourier's symbol of $D_{\pm}^{\alpha-1}$ is $(\mp \mathrm{i}k)^{\alpha-1}$ [31], the non-local part $\frac{\sin\frac{\pi}{2}(\alpha-\theta)}{\sin\pi\alpha}D_{+}^{\alpha-1} - \frac{\sin\frac{\pi}{2}(\alpha+\theta)}{\sin\pi\alpha}D_{-}^{\alpha-1}$ has Fourier's symbol $|k|^{\alpha-1}\,\mathrm{e}^{\mathrm{isgn(k)}(\theta+1)\pi/2}$, which [24] obtained for derivable functions, of course satisfying $D_{\pm}^{\mathrm{KG},\alpha-1}P(.,t) = 0$ identically.

In a semi-infinite domain limited by a reflective barrier, according to (2) the flux is equal to ${}_{\infty}\mathcal{W}_{l,\tau}^{\alpha,\theta}(x,t)P^{*} - Kl^{-\alpha}\int_0^{+\infty}((1-H)P^{*})(x-y,t)(F_{\alpha,\theta}^{-}(-y/l) - F_{\alpha,\theta}^{+}(y/l))\,\mathrm{d}y$. Due to the theorem, in the diffusive limit $l^{-\alpha}\int_0^{+\infty}(1-H)P^{*}(x-y,t)\big(F_{\alpha,\theta}^{-}(-y/l) - F_{\alpha,\theta}^{+}(y/l)\big)\,\mathrm{d}y$ tends to $(\lambda^{-}-\lambda^{+})D_{+}^{\alpha-1}((1-H)P^{*})(x)$. Hence, the macroscopic flux is

$$Q(x,t)P = K\left[\frac{\sin\frac{\pi}{2}(\alpha-\theta)}{\sin\pi\alpha}\big(D_{+}^{\alpha-1}P\big)(x,t) - \frac{\sin\frac{\pi}{2}(\alpha+\theta)}{\sin\pi\alpha}\big(D_{-}^{\alpha-1}P^{*}\big)(x,t)\right] + K\frac{\sin\frac{\pi}{2}(\alpha+\theta) - \sin\frac{\pi}{2}(\alpha-\theta)}{\sin\pi\alpha}D_{+}^{\alpha-1}((1-H)P)(x,t) - K\mathcal{I}_{\alpha,\theta}\Gamma(3-\alpha)\left[D_{+}^{\mathrm{KG},\alpha-1}P(x,t) + D_{-}^{\mathrm{KG},\alpha-1}P(x,t)\right], \tag{10}$$

also equal to

$$K\left(\frac{\sin\frac{\pi}{2}(\alpha-\theta)}{\sin\pi\alpha}\right)\partial_x\int_x^{+\infty}P(y)(y-x)^{2-\alpha}\,\mathrm{d}y + \frac{\sin\frac{\pi}{2}(\alpha+\theta)}{\sin\pi\alpha}\partial_x\int_{-\infty}^{x}P^{*}(y)(x-y)^{2-\alpha}\,\mathrm{d}y + K\left(\frac{\sin\frac{\pi}{2}(\alpha+\theta)}{\sin\pi\alpha} - \frac{\sin\frac{\pi}{2}(\alpha-\theta)}{\sin\pi\alpha}\right)\partial_x\int_{-\infty}^{0}P(-y)(x-y)^{2-\alpha}\,\mathrm{d}y - K\mathcal{I}_{\alpha,\theta}\Gamma(3-\alpha)\left[D_{+}^{\mathrm{KG},\alpha-1}P(x,t) + D_{-}^{\mathrm{KG},\alpha-1}P(x,t)\right], \tag{11}$$

provided P decreases to zero rapidly at infinity.

For $\alpha = 2$, the method has to be slightly adapted but still yields Fick's law. At the end of subsection 3.2, we pointed out that the case $\alpha = 2$ has to be considered separately. To do this, take $A(l)$ as a function, tending to $+\infty$ when l tends to zero, with $lA(l)$ tending to zero: for instance, we can choose $A(l) = l^{-1/2}$. Parameter θ is equal to zero, L_{α}^{0} is even and superscripts $\pm$ in $F_{2,0}^{\pm}$ are of no use: instead we put $F_{2,0}$. It is enough to consider the case of an infinite medium, since ${}_{0}\mathcal{W}_{l,\tau}^{\alpha,0}(x,t)P$ is equal to ${}_{\infty}\mathcal{W}_{l,\tau}^{\alpha,0}(x,t)P^{*}$. We have

$$l^{-2}\int_0^{+\infty}F_{2,0}\left(\frac{y}{l}\right)(P(x+y,t)-P(x,t))\,\mathrm{d}y = l^{-1}\int_0^{+\infty}F_{2,0}(y)(P(x+ly,t)-P(x,t))\,\mathrm{d}y$$
$$= \int_0^{A(l)}yF_{2,0}(y)\frac{P(x+ly,t)-P(x,t)}{ly}\,\mathrm{d}y + l^{-1}\int_{A(l)}^{+\infty}F_{2,0}(y)(P(x+ly,t)-P(x,t))\,\mathrm{d}y.$$

If $P(.,t)$ is differentiable at point x, $\int_0^{A(l)} F_{2,0}(y)\frac{P(x+ly,t)-P(x,t)}{ly}\,\mathrm{d}y$ tends to the usual derivative $\partial_x P(x,t)$, times $\int_0^{+\infty} F_{2,0}(y)y\,\mathrm{d}y$, itself equal to 1/2 due to $F_{2,0}(x) = \int_x^{+\infty}\frac{1}{2\sqrt{\pi}}\,\mathrm{e}^{-y^2/4}\,\mathrm{d}y$. And $l^{-1}\left|\int_{A(l)}^{+\infty} F_{2,0}(y)(P(x+ly,t)-P(x,t))\,\mathrm{d}y\right|$ is less than $l^{-2}F_{2,0}(A(l))\int_{lA(l)}^{+\infty} P(x+y,t)\,\mathrm{d}y+$ $P(x,t)l^{-1}\int_{A(l)}^{+\infty} F_{2,0}(y)\,\mathrm{d}y$, which tends to 0 when $P(.,t)$ is fixed in L^1. Similar results are obtained at the left of x; hence for $\alpha = 2$, in the limit 'l tends to zero' operator flux tends to $-K\partial_x P(x,t)$, in agreement with classical Fick's law.

The more general fractional version implies a space-fractional variant of the classical diffusion equation.

4.3. *Space-fractional diffusion equation*

When the density of particles and the macroscopic flux are derivable, mass conservation without sources implies $\partial_t P(x,t) = -\partial_x Q(x,t)$. Moreover, we have $\partial_x D_\pm^{\alpha-1} = \pm D_\pm^\alpha$, and local Kolwankar–Gangal derivatives with order of less than 1 are identically zero. Hence, in an infinite medium, we deduce from (8) and (9) that P evolves according to

$$\partial_t P(x,t) = -K\left[\frac{\sin\frac{\pi}{2}(\alpha-\theta)}{\sin\pi\alpha}D_+^\alpha P(x,t) + \frac{\sin\frac{\pi}{2}(\alpha+\theta)}{\sin\pi\alpha}D_-^\alpha P(x,t)\right]. \quad (12)$$

The right-hand side of (12) is equal to

$$-\frac{K}{\Gamma(2-\alpha)}\partial_{x^2}\left[\frac{\sin\frac{\pi}{2}(\alpha-\theta)}{\sin\pi\alpha}\int_{-\infty}^{x}(x-y)^{2-\alpha}P(y,t)\,\mathrm{d}y + \frac{\sin\frac{\pi}{2}(\alpha+\theta)}{\sin\pi\alpha}\int_x^{+\infty}(y-x)^{2-\alpha}P(y,t)\,\mathrm{d}y\right],$$

which is K times the Riesz–Feller derivative $\nabla_x^{\alpha,\theta}P$ of order α and skewness parameter θ [39, 43–45]. We thus retrieve the fractional dispersion equation which had been obtained via Fourier's analysis, from the generalized master equation for the density of particles performing possibly skewed Lévy flights.

In a medium, limited by a reflective barrier, we obtain

$$\partial_t P(x,t) = -K\left[\frac{\sin\frac{\pi}{2}(\alpha-\theta)}{\sin\pi\alpha}D_+^\alpha P^*(x,t) + \frac{\sin\frac{\pi}{2}(\alpha+\theta)}{\sin\pi\alpha}D_-^\alpha P(x,t)\right] - K\frac{\sin\frac{\pi}{2}(\alpha-\theta)-\sin\frac{\pi}{2}(\alpha+\theta)}{\sin\pi\alpha}D_+^\alpha((1-H)P^*)(x,t). \quad (13)$$

Also the right-hand side of (13) is

$$-\frac{K}{\Gamma(2-\alpha)}\partial_{x^2}\left[\frac{\sin\frac{\pi}{2}(\alpha-\theta)}{\sin\pi\alpha}\left[\int_0^x(x-y)^{2-\alpha}P(y,t)\,\mathrm{d}y + \int_x^{+\infty}(x+y)^{2-\alpha}P(y,t)\,\mathrm{d}y\right] + \frac{\sin\frac{\pi}{2}(\alpha+\theta)}{\sin\pi\alpha}\int_x^{+\infty}(y-x)^{2-\alpha}P(y,t)\,\mathrm{d}y - \frac{K}{\Gamma(2-\alpha)}\frac{\sin\frac{\pi}{2}(\alpha-\theta)-\sin\frac{\pi}{2}(\alpha+\theta)}{\sin\pi\alpha}\int_0^{\infty}(x+y)^{2-\alpha}P(y,t)\,\mathrm{d}y\right].$$

For symmetric random walks (with $\theta = 0$) we retrieve a result of [28, 29] with, on the right-hand side, $K\nabla_x^{\alpha,\theta}HP$, plus diffintegrals with kernel $(x+y)^{2-\alpha}$, which account for the influence of the wall. That (13) represents the evolution of the concentration of walkers had been checked in [29] by comparing the discretized solution of the partial differential equation (13) against Monte Carlo simulations of symmetric Lévy flights.

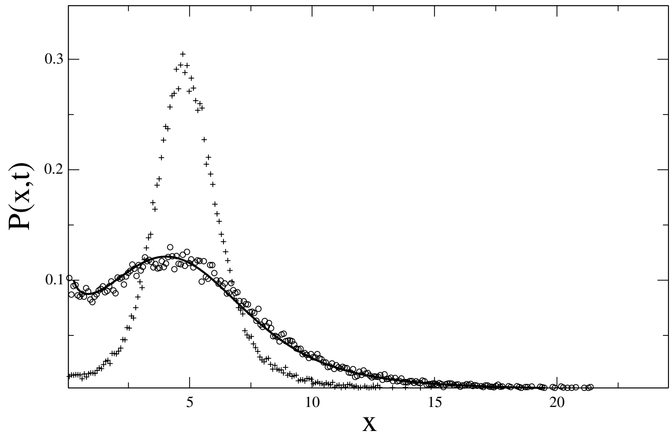

Figure 1. Numerical solution of (13), compared with the Monte Carlo simulation of skewed Lévy flights with a reflective barrier, for $\alpha = 1.5$, with $\theta = 0.2$. Full line represents the numerical solution to (13) with $K = 1$ at instant $t = 4$ and circles stand for random walk histograms. Histograms at time $t = 1$ correspond to symbols 'plus'.

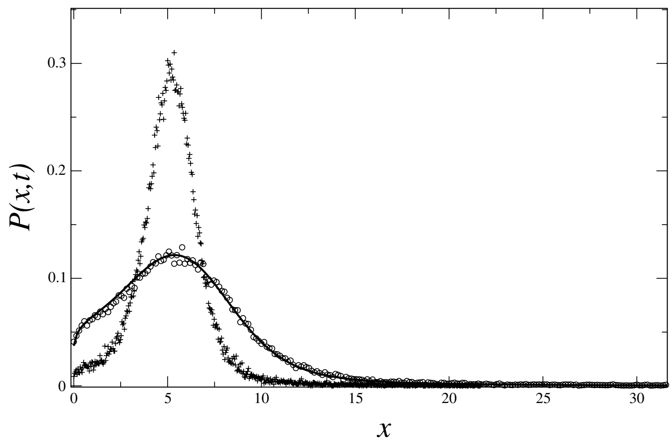

Figure 2. Numerical solution of (13), compared with the Monte Carlo simulation of skewed Lévy flights with a reflective barrier, for $\alpha = 1.5$, with $\theta = -0.2$. Full line represents the numerical solution to (13) with $K = 1$ at instant $t = 4$ and circles stand for random walk histograms. Histograms at time $t = 1$ correspond to symbols 'plus'.

4.4. Numerical illustration of (13)

In order to solve (13), derivatives of order α were discretized according to a shifted Grüunwald–Letnikov scheme [44, 45], and we have set $K = 1$, as in [29]. The issue was compared with histograms of Lévy flights corresponding to small values of τ and l satisfying $\tau = l^{\alpha}$. In order to keep coherent data, we had to take into account that Dirac impulses are easy to implement in Monte Carlo simulations, but not in the discretized partial differential equation (13). Hence, numerical simulations of (13) were started at time $t = 1$ from corresponding histograms, represented by symbols 'plus' in figures 1 and 2. Random walks were started at $t = 0$ from Dirac impulses applied at $x = 5$.

We observe that already at instant $t = 1$ the maximum of the density of particles has moved from initial impulse's location $x = 5$ to the left for $\theta = 0.2$, and to the right for $\theta = -0.2$. The trend is confirmed at larger values of t, but for positive-valued θ the peak of the distribution of particles is perturbed by the wall (at $x = 0$).

5. Conclusion

For a cloud of particles, performing one-dimensional Lévy flights with time and length scales τ and l satisfying $\tau = Kl^{\alpha}$, the mass flux through the abscissa x is a difference between convolutions involving the density of walkers. Convolutions have kernels, which are cumulated probabilities $\mathbf{P}\{Y > y\}$ and $\mathbf{P}\{Y < -y\}$ for a jump to be directed to the right or to the left and to have an amplitude $|Y|$ larger than y. Hence, they behave asymptotically as $y^{-\alpha}$, with α being the stability exponent of the considered Lévy flights. Kernels also incorporate dilatation by $1/l$ and amplification by $l^{-\alpha}$. We showed that, in the limit when l tends to zero, convolutions of this form tend to fractional derivatives of order $\alpha - 1$, provided kernels have integrals (computed over $]-\infty, 0]$ and $[0, +\infty[$) equal to zero. Kernels, present in the flux, do not match this condition, but we could use the result by cutting the convolutions into two parts, local and non-local. That distribution functions of stable Lévy laws on the left and on the right have equal integrals over $]-\infty, 0]$ and $[0, +\infty[$, even when they are skewed was essential for that.

This way, we showed that, in the diffusive limit and within the context of Lévy flights, the mass flux is a combination of fractional derivatives of the concentration of walkers. We have usual derivatives of order $\alpha - 1$, defined by integrals over half-lines, at the left and at the right of point x where the flux is considered. We also have local Kolwankar–Gangal's derivatives, which are only visible at points, if there are, where the density of particles cannot be represented by a function, possessing derivatives of order larger than $\alpha - 1$. Such a possibility may occur at points where sources or sinks are applied, or represent absorbing or adsorbing boundaries. Hence, a fractional generalization of Fick's law was derived, without passing through any partial differential equation for the time evolution of the concentration.

The thus obtained fractional Fick's law, when recast into mass conservation principle, yields such a partial differential equation, provided local derivatives are zero, and in fact the condition was satisfied in all circumstances where space-fractional dispersion equations were previously derived. That the method applies to situations with boundaries, so that Fourier's analysis is not simple, was illustrated by a non-trivial example.

Appendix A. Densities of alpha stable Lévy laws

Stable laws generalize Gaussian statistics [33, 35]. In many occasions, the word 'stable' refers to some property, invariant under a definite set of transformations, and here we consider that dilatations and translations do not affect qualitatively a given probability law. We use the word 'stable' for laws, which keep similar under the addition of independent identical random variables, up to dilatations and translations. For precise definitions, we refer to [33, 35].

When the random variable X is stable, with law F, for any sequence of independent random variables X_i distributed like X, there exists a sequence c_n of positive numbers such that $\frac{X_1+\cdots+X_n}{c_n}$ be distributed according to F itself for any positive integer n [33, 35]. Moreover, c_n is a power of n, and the inverse α of the exponent belongs to $]0, 2]$ and serves as a label for the law: it is called the stability exponent of the α stable law. The value $\alpha = 2$ corresponds to normal law, which is symmetric. For $\alpha \in]0, 2[$, stable laws may be symmetric or skewed.

Stable laws play an important role in Nature because they are attractors, again in the context of the addition of many independent random variables X_n, distributed according to law F. Probability law G is an attractor for F if there exist sequences A_n and B_n, with $B_n > 0$, such that the law of $\frac{X_1+\cdots+X_n}{B_n} - A_n$ tends to G when n tends to ∞ [33].

Loosely speaking, α-stable laws are attractors for probability laws whose density behaves asymptotically as $x^{-\alpha-1}$ if α belongs to $]0, 2[$, normal law (with $\alpha = 2$) is an attractor for probability laws whose asymptotics is $x^{-\gamma-1}$ with $\gamma \geqslant 2$ [33, 35].

Except for some values (e.g. $\alpha = 1$ or 2), the density of a stable law cannot be given in closed form. But, up to translations and dilatations, the Fourier transform is $e^{-|k|^\alpha e^{\mathrm{isign}(k)\pi\theta/2}}$. Hence, the corresponding density L_α^θ satisfies $L_\alpha^\theta(-x) = L_\alpha^{-\theta}(x)$. It is labelled by the stability exponent α, and the skewness parameter θ, which satisfies $|\theta| \leqslant \min(\alpha, 2-\alpha)$, and hence belongs to $[\alpha-2, 2-\alpha]$ for α in $]1, 2]$.

In neighbourhoods of $+\infty$, except for $\alpha = 2$, L_α^θ behaves as a negative power of the variable [39, 46]. For α strictly between 1 and 2 with $\alpha - 2 < \theta \leqslant 2-\alpha$, provided $x > A > 0$ holds with A large enough, we have

$$L_\alpha^\theta(x) = \frac{1}{\pi x}\Sigma_{n=1}^{+\infty}(-x^{-\alpha})^n\frac{\Gamma(1+n\alpha)}{n!}\sin\frac{n\pi}{2}(\theta-\alpha). \tag{A.1}$$

We will denote by

$$C_\alpha^\theta = \frac{-1}{\pi}\Gamma(1+\alpha)\sin\frac{\pi}{2}(\theta-\alpha) \tag{A.2}$$

the coefficient of the leading term in expansion (A.1).

Appendix B. Riemann–Liouville fractional integrals and derivatives

The definition and essential properties are in [31, 40]. Since notations are not universal, we recall those which we use here. Let α' be positive: the left-sided fractional integral of order α', computed over $[a, x]$, is

$$I_{a,+}^{\alpha'}\varphi(x) = \frac{1}{\Gamma(\alpha')}\int_a^x (x-y)^{\alpha'-1}\varphi(y)\,\mathrm{d}y \tag{B.1}$$

according to [31]. Here we focus on the case $a = -\infty$, with the simplified notation $I_+^{\alpha'}\varphi(x) = I_{-\infty,+}^{\alpha'}\varphi(x)$ of [40]. Right-sided integrals are

$$I_{b,-}^{\alpha'}\varphi(x) = \frac{1}{\Gamma(\alpha')}\int_x^b (y-x)^{\alpha'-1}\varphi(y)\,\mathrm{d}y \tag{B.2}$$

with $I_-^{\alpha'}\varphi(x) = I_{+\infty,-}^{\alpha'}\varphi(x)$.

The corresponding left-sided Riemann–Liouville derivative of order α' is

$$\mathcal{D}_+^{\alpha'}\varphi(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{[\alpha']+1} I_+^{1-\{\alpha'\}} = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{[\alpha']+1}\frac{1}{\Gamma([\alpha']+1-\alpha')}\int_{-\infty}^x (x-y)^{-\{\alpha'\}}\varphi(y)\,\mathrm{d}y, \tag{B.3}$$

where $[.]$ denotes integer part, while $\{.\}$ is defined by $\alpha' = [\alpha'] + \{\alpha'\}$. The right-sided Riemann–Liouville derivative is

$$\begin{aligned}\mathcal{D}_-^{\alpha'}\varphi(x) &= \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^{[\alpha']+1} I_-^{1-\{\alpha'\}} \\ &= \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^{[\alpha']+1}\frac{1}{\Gamma([\alpha']+1-\alpha')}\int_x^{+\infty}(y-x)^{-\{\alpha'\}}\varphi(y)\,\mathrm{d}y.\end{aligned} \tag{B.4}$$

When α' is a positive integer, $\mathcal{D}_-^{\alpha'}$ and $\mathcal{D}_+^{\alpha'}$ are usual right and left-sided derivatives or order α'. A natural question is whether fractional derivatives defined by (B.3) and (B.4) share with derivatives of integer order the property of being left inverses to the corresponding integrals. In fact, when φ is in $L^1_{\mathrm{loc}}(R)$, if moreover integrals $I_\pm^{[\alpha']+1}$ are absolutely convergent, we have $\left(\mathcal{D}_\pm^{\alpha'} I_\pm^{\alpha'} \varphi\right)(x) = \varphi(x)$ a.e., due to lemma 4.7 in [40]. When the above hypotheses are satisfied, $\mathcal{D}_\pm^{\alpha'}$ can be thought of as being a left inverse to $I_\pm^{\alpha'}$, which fails to hold if $I_\pm^1 \varphi$ and $I_\pm^{[\alpha']+1}\varphi$ do not belong to $L^1(R)$, even for φ in $L^p(R)$ with $1 < p < 1/\alpha'$. Marchaud's definition seems to give a more appropriate left inverse to fractional integrals.

Marchaud's definition combines generalized finite differences and fractional integrals. A rather general definition of finite differences Δ_t^n was used by [40], in view of complex orders of derivation. Here we only need real-valued orders, and it is enough to take a definition connected with translations T_t of amplitude t. Here, the definition of Δ_t^n is $(Id - T_t)^n f(x) = \Sigma_{j=0}^n \binom{n}{j}(-1)^j f(x - jt)$, which becomes $f(x-t) - f(x)$ for $n = 1$. With these notations, Marchaud's derivative $D_\pm^{\alpha'}$ of function f is the limit, when $\varepsilon \to 0+$ of

$$D_{\pm,\varepsilon}^{\alpha'} f(x) = \frac{1}{\int_0^{+\infty} t^{-\alpha'-1}(1-\mathrm{e}^{-t})^n \,\mathrm{d}t} \int_\varepsilon^{+\infty} t^{-\alpha'-1} \Delta_{\pm t}^n f(x)\,\mathrm{d}t, \tag{B.5}$$

with $n > \alpha$. For $0 < \alpha' < 1$, we have $n = 1$ and (B.5) becomes $D_{\pm,\varepsilon}^{\alpha'} f(x) = \frac{-1}{\Gamma(-\alpha')} \int_\varepsilon^{+\infty} t^{-\alpha'-1}[f(x) - f(x \mp t)]\,\mathrm{d}t$. For $\alpha' = 1$, we have to put $n = 2$ in (B.5) if we want to use this expression, but we can also consider that $D_\pm^{\alpha'}$ is the usual left- or right-sided derivative of order α' when α' is a non-negative integer.

We thus have a left inverse for $I_\pm^{\alpha'}$ in a wider domain, which in some sense is optimal, since it provides a characterization of $I_\pm^{\alpha'} L^p$ for $1 < p < 1/\alpha'$. Indeed, for $0 < \alpha' < 1$, theorem 6.2 of [31] states the following: if the $L^p(R)$ limit of $D_{\pm,\varepsilon}^{\alpha'} f$ exists when $\varepsilon \to 0+$, or if $\sup_{\varepsilon>0} \left\| D_{\pm,\varepsilon}^{\alpha'} f \right\|_{L^p(R)}$ is finite, if moreover, f belongs to $L^p(R)$ with $1 \leqslant r < \infty$, then f belongs to $I_\pm^{\alpha'} L^p(R)$ and there exists φ s.t. $f(x) = I_\pm^{\alpha'}\varphi(x)$ almost everywhere in R. The theorem was stated in $L^p(R)$, but the proof adapts without any modification to $L^p]-\infty, a]$ for $D_+^{\alpha'}$ and to $L^p[a, +\infty[$ for $D_-^{\alpha'}$. Derivatives $D_\pm^{\alpha'}$ and $\mathcal{D}_\pm^{\alpha'}$ coincide for functions of the form $I_\pm^{\alpha'}\varphi$ with φ in L^1_{loc} such that $I_\pm^{[\alpha']+1}$ converges absolutely [40].

Other expressions yield the left inverse of $I_\pm^{\alpha'}$. Among them, the Grüunwald–Letnikov fractional derivative [31] or order α' of f is the limit, when mesh h tends to zero, of $h^{-\alpha'}$ times the series $\Sigma_{k=0}^\infty (-1)^k \binom{\alpha'}{k} f(x - kh)$. It provides useful approximations to Riemann–Liouville and Marchaud's derivatives, connected with finite differences numerical schemes.

Appendix C. Kolwankar and Gangal's local fractional derivatives

The notion of a local fractional derivative was introduced [41] in view of building a tool, designed for the study of continuous but nowhere differentiable functions frequently occurring in Nature and economics. Those fractional derivatives share some properties with previously defined ones, such as chain rule or generalized Leibniz rule [47]. They are very useful for to compute fractal dimensions of graphs. In fact, we will see that they vanish for smooth enough functions, and hence can become 'invisible'.

For q between 0 and 1, the right-sided Kolwankar and Gangal's [41] fractional derivative of order q of function f, computed at x, is

$$D_+^{\mathrm{KG},q} f(x) = \lim_{h\to 0+} \frac{\mathrm{d}}{\mathrm{d}h} I_{x,+}^{1-q}(f(.) - f(x))(x+h).$$

The definition makes sense when f is continuous in $[x, x+\varepsilon]$, while moreover function $I_{x,+}^{1-q}(f(.) - f(x))(x+h)$ (of the variable h) is derivable in $[0, \varepsilon]$, with positive ε. When the above limit exists, it is also equal to the limit, when h tends to 0+, of $h^{-1}I_{x,+}^{1-q}(f(.)-f(x))(x+h)$, due to l'Hôpital's rule. Indeed, we have $h^{-1}I_{x,+}^{1-q}(f(.)-f(x))(x+h) = \frac{h^{-q}}{\Gamma(1-q)}\int_0^1(1-t)^{-q}(f(x+th)-f(x))\,\mathrm{d}t$. Note that if f is continuous in $[x, x+\varepsilon]$, with $\lim_{h\to 0+}\frac{f(x+h)-f(x)}{h^q} = a$, this implies $D_+^{\mathrm{KG},q}f(x) = a$.

At the left, similarly we have

$$D_-^{\mathrm{KG},q}f(x) = \lim_{h\to 0+}\frac{\mathrm{d}}{\mathrm{d}h}I_{x,-}^{1-q}(f(x)-f(.))(x-h),$$

also equal to the limit, when h tends to 0+, of $h^{-1}I_{x,-}^{1-q}(f(x)-f(.))(x-h)$. If, for positive and finite $b-a$ and with $q < q+\varepsilon < 1$, the function f belongs to Hölder space $H^{q+\varepsilon}[a,b]$, so that $\sup_{x,y\in[a,b]}\left(\frac{|f(y)-f(x)|}{|y-x|^{q+\varepsilon}}\right)$ is finite, we immediately see that Kolwankar and Gangal's local derivatives of order q are zero in $[a,b]$. Functions f that are derivable everywhere on an interval, except at the right of point x_0 (in the interior or at a boundary) where they behave as $a(x-x_0)^q$, have a right-sided Kolwankar and Gangal's derivative everywhere: it is zero, except at x_0+, where it is equal to a times a constant.

Appendix D. Proof of the lemma

Lemma. *If F satisfies H_1 and $H_2(\alpha)$, with $1 < \alpha < 2$, then $\int_0^s F(T)(s-T)^{\alpha-2}\,\mathrm{d}T$ is integrable in R^+.*

Proof. Set $\alpha' = \alpha - 1$. If F is as F_1 in hypothesis $H_2(\alpha)$'s statement, lemma 4.12 of [40] shows that

$$\Gamma(\alpha')I_-^{\alpha'}(HF)(s) = \int_0^s F(T)(s-T)^{\alpha'-1}\,\mathrm{d}T$$

is in L^1. Hence, it is enough to prove the present lemma for $F = -\frac{1}{\alpha'}\chi_{[0,1]} + x^{-\alpha'-1}\chi_{[1,+\infty[}$, since modifying F_1 will immediately lead to the general case. Then, we have

$$\int_0^x (x-y)^{\alpha'-1}\chi_{[0,1]}(y)\,\mathrm{d}y = \frac{x^{\alpha'}-(x-1)^{\alpha'}}{\alpha'}$$

for $x > 1$, and

$$\int_0^x (x-y)^{\alpha'-1}y^{-\alpha'-1}\chi_{[1,+\infty[}(y)\,\mathrm{d}y = x^{-1}\left(G(1)-G(1/x)+\frac{x^{\alpha'}-1}{\alpha'}\right)$$

when x is large enough, with G being defined by $G(X) = \int_0^X[(1-z)^{\alpha'-1}-1]z^{-\alpha'-1}\,\mathrm{d}z$. From this we deduce

$$\int_0^x F(t)(x-t)^{\alpha'-1}\,\mathrm{d}t = \alpha'^{-1}(x^{\alpha'-1}-\alpha'^{-1}x^{\alpha'}(1-(1-1/x)^{\alpha'})) + x^{-1}(G(1)-\alpha'^{-1}) - x^{-1}G(1/x). \tag{D.1}$$

□

Function $\frac{(1-t)^{\alpha'-1}-1}{t}$ is continuous and integrable in $[0,1[$. In the neighbourhood of 0, $\frac{(1-t)^{\alpha'-1}-1}{t}t^{-\alpha'}$ is equivalent to $(1-\alpha')t^{-\alpha'}$, hence $G(1/x)$ is equivalent to $x^{\alpha'-1}$ when x is large. Hence $x^{-1}G(1/x)$ is integrable in a neighbourhood of $+\infty$. It is also the case for $\alpha'^{-1}[x^{\alpha'-1}-\alpha'^{-1}x^{\alpha'}(1-(1-1/x)^{\alpha'})$. We now will check that $G(1)-\alpha'^{-1}$ is zero. To see this, set $g(p,q)=\int_0^1((1-t)^{q-1}-1)t^{p-1}\,\mathrm{d}t$. For complex-valued p and q satisfying $\mathrm{Re}(p)>0$ and $\mathrm{Re}(q)>0$, $\int_0^1(1-t)^{q-1}t^{p-1}\,\mathrm{d}t$ is a Bernoulli beta function [48] and we have

$$g(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}-\frac{1}{p}. \tag{D.2}$$

Let us fix $q=\alpha'$ and vary the complex number p: t^p is a function of p, whose derivative $t^pLn(t)$ is dominated by the $L^1]0,1[$ function $t^p|Ln(t)|$ for $\mathrm{Re}(p)\geqslant p_0>-1$, so that, by dominated convergence, $g(p,\alpha')$ is derivable with respect to p. Hence it is analytic for $\mathrm{Re}(p)\geqslant p_0>-1$. Since $\frac{\Gamma(q)}{\Gamma(p+q)}$ is also analytic in the neighbourhood of 0 while $\Gamma(p)$ has a simple pole with residue 1, the right-hand side of (D.2) is holomorphic for $\mathrm{Re}(p)\geqslant p_0>-1$. Hence relation (D.2) holds for $p=-\alpha'$, and the lemma is proved.

Appendix E. Integrals of cumulated alpha stable Lévy laws

Due to symmetry, integrals $\int_0^{+\infty}F^+_{\alpha,\theta}(y)\,\mathrm{d}y$ and $\int_0^{+\infty}F^-_{\alpha,\theta}(-y)\,\mathrm{d}y$ are equal for $\theta=0$. In fact, and this point is essential for us, this equality holds for all admissible values of θ. Let us prove the claim.

First, note that $F^-_{\alpha,\theta}(-x)=\int_{-\infty}^{-x}L^\theta_\alpha(y)\,\mathrm{d}y=\int_x^{+\infty}L^{-\theta}_\alpha(-y)\,\mathrm{d}y=F^+_{\alpha,-\theta}(x)$. Then, we will uses Mellin's transform, defined by $\mathcal{M}\omega(z)=\int_0^{+\infty}t^{z-1}\omega(t)\,\mathrm{d}t$ for the function ω. With $z=1$ we see that $\int_0^{+\infty}F^+_{\alpha,\theta}(y)\,\mathrm{d}y=\mathcal{M}F^+_{\alpha,\theta}(1)$, while we have $F^+_{\alpha,\theta}(x)=I^1_-L^{-\theta}_\alpha(x)$, hence $\int_0^{+\infty}F^+_{\alpha,\theta}(y)\,\mathrm{d}y=\left(\mathcal{M}I^1_-L^{-\theta}_\alpha\right)(1)$.

For $z>1$ and sufficiently good-behaved functions in neighbourhoods of ∞, such as L^θ_α, we have

$$\left(\mathcal{M}I^1_-\omega\right)(z)=\frac{\Gamma(z)}{\Gamma(z+1)}(\mathcal{M}\omega)(z+1),$$

for $z<\alpha$ according to [40, p 44]. From this, due to $F^+_{\alpha,\theta}(x)=\int_x^{+\infty}L^\theta_\alpha(y)\,\mathrm{d}y=I^1_-L^\theta_\alpha(x)$, we deduce

$$\left(\mathcal{M}F^d_{\alpha,\theta}\right)(z)=\frac{\Gamma(z)}{\Gamma(z+1)}\left(\mathcal{M}L^\theta_\alpha\right)(z+1).$$

The Mellin transform $\mathcal{M}L^\theta_\alpha$ is given in [46]

$$\left(\mathcal{M}L^\theta_\alpha\right)(z)=\frac{1}{\alpha}\frac{\Gamma(z)\Gamma((1-z)\alpha^{-1})}{\Gamma\left((1-z)\frac{\alpha-\theta}{2\alpha}\right)\Gamma\left(1-(1-z)\frac{\alpha-\theta}{2\alpha}\right)},$$

which is of the form

$$\left(\mathcal{M}L^\theta_\alpha\right)(z)=\frac{1}{\pi\alpha}\Gamma(z)\Gamma\left(\frac{1-z}{\alpha}\right)\sin\left((1-z)\pi\frac{\alpha-\theta}{2\alpha}\right) \tag{E.1}$$

due to complements formula for Gamma functions [48]. In fact, (E.1) for $0<\mathrm{Re}(z)<1$ has been proved in [46]. Nevertheless, $\mathcal{M}L^\theta_\alpha(z)$, as a function of z, is holomorphic for $0<\mathrm{Re}(z)<\alpha+1$, due to the behaviour of $L^\theta_\alpha(x)$ for large real values of x. On the right-hand side of (E.1), $\Gamma(z)\Gamma((1-z)\alpha^{-1})$ is holomorphic also except at poles of $\Gamma((1-z)\alpha^{-1})$, which means that we have to exclude 1 from $\{z\in C/0<\mathrm{Re}(z)<\alpha+1\}$. Then, analytic continuation extends (E.1) to $\{z\in C/0<\mathrm{Re}(z)<\alpha+1\}-\{1\}$.

From this we deduce $\int_0^{+\infty}F^+_{\alpha,\theta}(y)\,\mathrm{d}y=\left(\mathcal{M}F^+_{\alpha,\theta}\right)(2)$

$$=\frac{\Gamma(2)\Gamma(-1/\alpha)}{\alpha\pi}\sin\pi\frac{\theta-\alpha}{2\alpha}=-\frac{\Gamma(-1/\alpha)}{\alpha\pi}\cos\pi\frac{\theta}{2\alpha}. \tag{E.2}$$

We see that $\int_0^{+\infty}F^+_{\alpha,\theta}(y)\,\mathrm{d}y$ is an even function of θ, hence the claimed result.

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